Sydney Girls High School
2020

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions	• Reading time – 10 minutes • Working time – 3 hours • Write using a black pen • Calculators approved by NESA may be used • A reference sheet is provided • In Questions 11-16, show relevant mathematical reasoning and/or calculations
Total marks: 100	**Section I – 10 marks (pages 3-7)** • Attempt Questions 1-10 • Allow about 15 minutes for this section **Section II – 90 marks (pages 8-15)** • Attempt Questions 11-16 • Allow about 2 hours and 45 minutes for this section

Name: …………………………………………………… Teacher: ……………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2020 HSC Examination Paper in this subject.

Section I

10 marks

Attempt Questions 1–10

Use the multiple-choice answer sheet for Questions 1–10.

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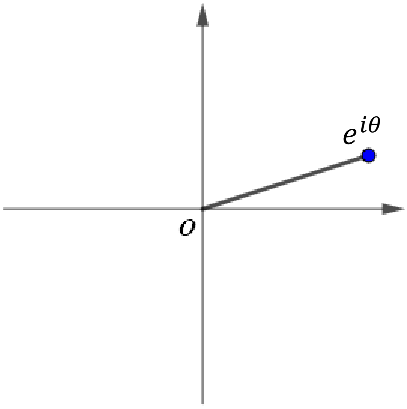
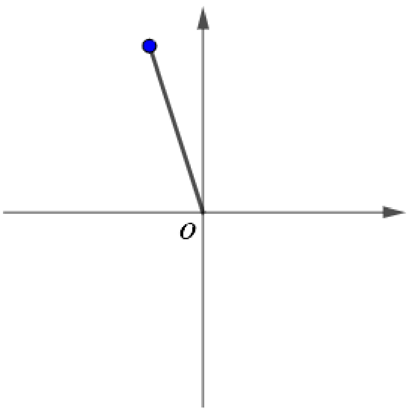
(1) The Argand diagram shows the complex number $e^{i\theta}$.

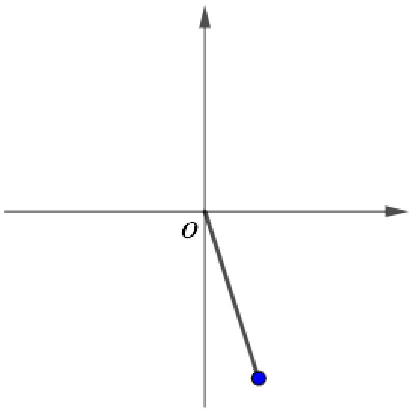


Which of the following diagrams best shows the complex number $-ie^{2i\theta}$?

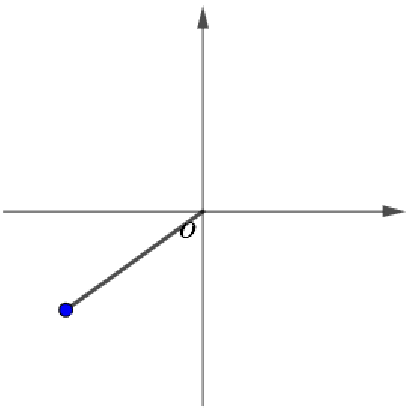
(A)



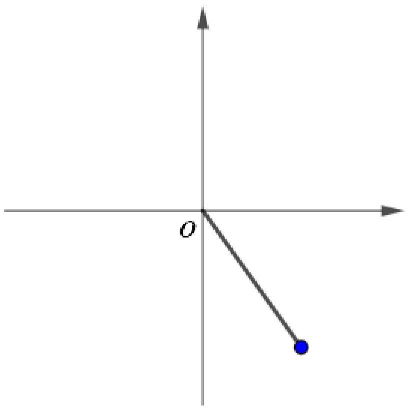
(B)



(C)



(D)



(2) A body of mass 5 kg is acted upon by a variable force $F = 15(t^2 + 3)$ Newtons, where t is in seconds. If the body starts from rest, which of the following is the velocity function?

(A) $v = t^3 + 9t$

(B) $v = 5t^3 + 45t$

(C) $v = \sqrt{2t^3 + 54t}$

(D) $v = \sqrt{10t^3 + 270t}$

(3) Which pair of vectors are perpendicular?

(A) $2\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$ and $4\underset{\sim}{i} - 25\underset{\sim}{j} + 9\underset{\sim}{k}$

(B) $\underset{\sim}{i} - 21\underset{\sim}{j} + 12\underset{\sim}{k}$ and $3\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$

(C) $5\underset{\sim}{i} - \underset{\sim}{j} + 17\underset{\sim}{k}$ and $2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

(D) $-3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $2\underset{\sim}{i} + 3\underset{\sim}{j} + 5\underset{\sim}{k}$

(4) Which of the following statements is **false** for real values of x and y?

(A) $\forall x, \forall y: x^2 > y + 1.$

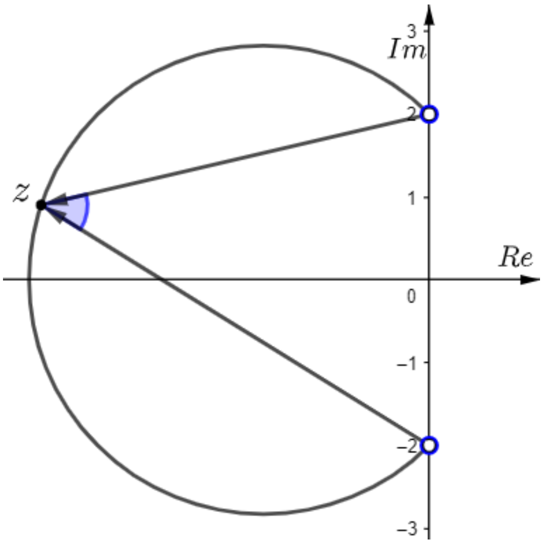
(B) $\forall x, \exists y: x^2 < y + 1.$

(C) $\exists x, \forall y: x^2 > y + 1.$

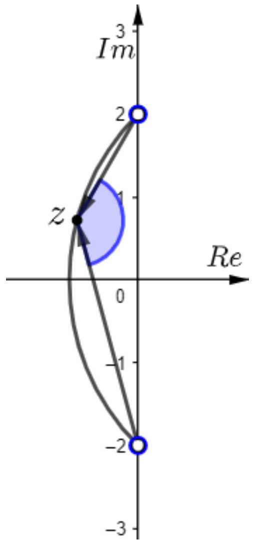
(D) $\exists x, \exists y: x^2 < y + 1.$

(5) Which diagram represents z such that $\arg\left(\frac{z+2i}{z-2i}\right) = \frac{3\pi}{4}$?

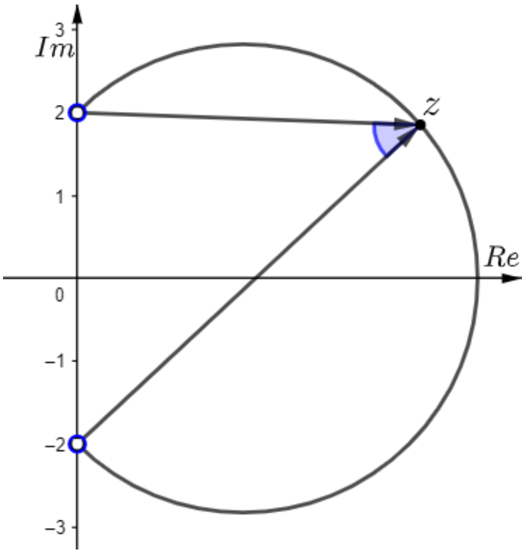
(A)



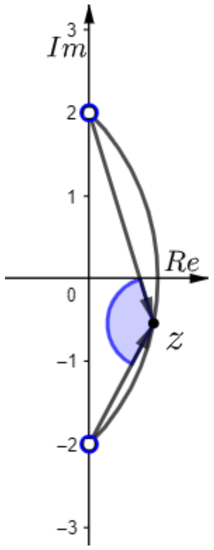
(B)



(C)



(D)



(6) What is the derivative of $\sin^{-1} x - \sqrt{1-x^2}$?

(A) $\frac{\sqrt{1+x}}{\sqrt{1-x}}$

(B) $\frac{\sqrt{1+x}}{1-x}$

(C) $\frac{1+x}{\sqrt{1-x}}$

(D) $\frac{1+x}{1-x}$

(7) The fifth roots of $1+\sqrt{3}i$ are:

(A) $\sqrt[5]{2}e^{-\frac{4\pi}{5}i}, \sqrt[5]{2}e^{-\frac{2\pi}{5}i}, \sqrt[5]{2}, \sqrt[5]{2}e^{\frac{2\pi}{5}i}, \sqrt[5]{2}e^{\frac{4\pi}{5}i}$

(B) $2e^{-\frac{4\pi}{5}i}, 2e^{-\frac{2\pi}{5}i}, 2, 2e^{\frac{2\pi}{5}i}, 2e^{\frac{4\pi}{5}i}$

(C) $\sqrt[5]{2}e^{-\frac{13\pi}{15}i}, \sqrt[5]{2}e^{-\frac{7\pi}{15}i}, \sqrt[5]{2}e^{-\frac{\pi}{15}i}, \sqrt[5]{2}e^{\frac{\pi}{3}i}, \sqrt[5]{2}e^{\frac{11\pi}{15}i}$

(D) $\sqrt[5]{2}e^{-\frac{11\pi}{15}i}, \sqrt[5]{2}e^{-\frac{\pi}{3}i}, \sqrt[5]{2}e^{\frac{\pi}{15}i}, \sqrt[5]{2}e^{\frac{7\pi}{15}i}, \sqrt[5]{2}e^{\frac{13\pi}{15}i}$

(8) A particle moves in a straight line so that its acceleration at any time is given by $\ddot{x} = -4x$. What is the period and amplitude given that initially $x = 3$ and $v = -6\sqrt{3}$?

(A) $T = \frac{\pi}{2}$ and $a = 3$

(B) $T = \frac{\pi}{2}$ and $a = 6$

(C) $T = \pi$ and $a = 3$

(D) $T = \pi$ and $a = 6$

(9) Which of the following is an expression for $\int \sin^2 x \cos^5 x \, dx$?

(A) $-\frac{\sin^3 x}{3} + \frac{2\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$

(B) $-\sin^3 x + 2\sin^5 x - \sin^7 x + c$

(C) $\frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + c$

(D) $\sin^3 x - 2\sin^5 x + \sin^7 x + c$

(10) Which of the following is an expression for $\int x^4 \log_e x \, dx$?

(A) $\frac{x^4 \log_e x}{4} - \frac{x^5}{25} + c$

(B) $\frac{x^4 \log_e x}{4} - \frac{x^5}{5} + c$

(C) $\frac{x^5 \log_e x}{5} - \frac{x^5}{25} + c$

(D) $\frac{x^5 \log_e x}{5} - \frac{x^5}{5} + c$

Section II

90 marks

Attempt Questions 11–16

Start each question on a NEW sheet of paper.

Question 11 (15 marks)

(a) If $z = 2 - i\sqrt{12}$

(i) Express z in modulus-argument form. [2]

(ii) Find the modulus and argument of z^5. [2]

(b) Find

(i) [2]

$$\int \frac{2}{\sqrt{4-9x^2}}\,dx$$

(ii) [2]

$$\int \frac{2x}{\sqrt{4-9x^2}}\,dx$$

(c)

(i) Express $\frac{8-x}{x(x-2)^2}$ in the form $\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$,
where A, B and C are constants. [2]

(ii) Hence find [2]

$$\int \frac{8-x}{x(x-2)^2}\,dx$$

(d) If ω is a complex root of the equation $z^3 - 1 = 0$.

(i) Show that $1 + \omega + \omega^2 = 0$. [1]

(ii) Prove that $(a+b)(a+\omega b)(a+\omega^2 b) = a^3 + b^3$. [2]

End of Question 11

Question 12 (15 marks)

Use a NEW sheet of paper.

(a) It is given that $|z+2| < \frac{1}{3}$, show that $|6z+11| \le 3$. [2]

(b) Sketch the region $\mathrm{Re}(z) \ge |z-\bar{z}|^2$. [3]

(c)

(i) Show that an equation of the line that goes through the points $(8,-19,13)$ and $(7,-15,10)$ is [2]

$$\underset{\sim}{r} = (8-\lambda)\underset{\sim}{i} - (19-4\lambda)\underset{\sim}{j} + (13-3\lambda)\underset{\sim}{k}$$

(ii) Take an interval on the line $\underset{\sim}{r}$ such that $-3 \le \lambda \le 7$ and find a point that divides the interval internally into a ratio of 3:2. [2]

(d)

(i) Explain why a cubic polynomial equation always has a real root. [1]

(ii) The cubic equation $x^3+bx^2+cx+d=0$ has a pure imaginary root. If the coefficients are real show that $d=bc$ and $c>0$. [2]

(e) If $|z-2i|=1$, find the greatest value of $|z-3|$. [3]

End of Question 12

Question 13 (15 marks)

Use a NEW sheet of paper.

(a) Numbers such as 6 and 28 are known as perfect numbers because they are equal to the sum of their factors, excluding the number itself.

A conjecture has been proposed that: if p is a perfect number then any multiple of p is also a perfect number.

(i) Use a counterexample to disprove this conjecture. [1]

(ii) Prove that: if p is a perfect number then no multiple of p is a perfect number. [2]

(b)

(i) Write down the greatest and least values of the expression [1]

$$\frac{1}{5+3\cos x}$$

(ii) Show that [2]

$$\frac{\pi}{16} \leq \int_0^{\frac{\pi}{2}} \frac{dx}{5+3\cos x} \leq \frac{\pi}{4}$$

(iii) Use the t-formulae to evaluate, correct to 3 decimal places, [3]

$$\int_0^{\pi/2} \frac{dx}{5+3\cos x}$$

(c)

(i) Given that $z = \cos\theta + i\sin\theta$, show that $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$. [1]

(ii) Express $\sin^5\theta$ in terms of multiples of θ. [3]

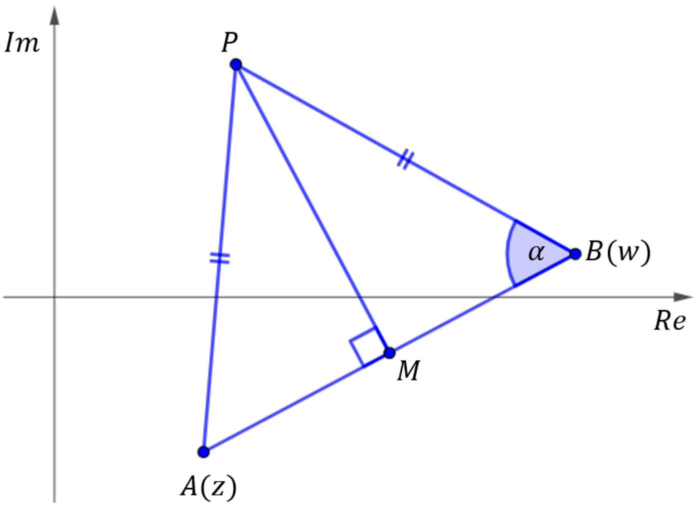
(iii) Hence, find $\int \sin^5\theta \, d\theta$. [2]

End of Question 13

Question 14 (15 marks)

Use a NEW sheet of paper.

(a) Triangle APB is isosceles with $PA = PB$ and $\angle ABP = \alpha$. Points A and B are represented by the complex numbers z and w repectively and M is the midpoint of AB.



(i) Explain why the distance $MP = \frac{1}{2}|w - z| \tan \alpha$. [1]

(ii) Show that the vector $\overrightarrow{MP} = \frac{1}{2} i(w - z) \tan \alpha$. [2]

(iii) If $\alpha = 45°$ show that the complex representation of the point P is [2]

$$\frac{1}{2}(w + iw + z - iz).$$

(b) Use mathematical induction to prove that the following is true for every integer $n \geq 2$, [3]

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$$

(c) If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the roots of the equation [2]

$$x^3 - (a+1)x^2 + (c-a)x - c = 0,$$

show that $\alpha + \beta + \gamma = n\pi + \frac{\pi}{4}$, where n is an integer.

(d) Consider the line $\underset{\sim}{r} = \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(i) Using the method of vector projections, show that the position vector of the point on the line $\underset{\sim}{r}$ closest to the point (x_0, y_0, z_0) is [2]

$$\frac{x_0 + 2y_0 + z_0}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(ii) Hence, find the point on the line $\underset{\sim}{r}$ that is closest to a second line [3]

$$\underset{\sim}{c} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}, \text{ where } t \in (-\infty, \infty).$$

End of Question 14

Question 15 (15 marks)

Use a NEW sheet of paper.

(a)

(i) Given $f(x) = f(a - x)$ and using the substitution $u = a - x$, prove that [2]

$$\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$$

(ii) Hence, or otherwise, evaluate in exact form: [2]

$$\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\,dx$$

(b) A particle P of mass 3 kg has simple harmonic motion in the x-direction described by the equation $\dot{x}^2 = 25\pi^2 - \pi^2 x^2$, where x is in metres.

(i) Show that $x = 5\cos(\pi t)$, where t is in seconds, is a solution to the equation. [1]

(ii) The particle is also undergoing simple harmonic motion in the y-direction such that $y = 5\sin(\pi t)$. Hence, the position of the particle can be represented in vector form by,

<u>Position</u>

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5\cos(\pi t) \\ 5\sin(\pi t) \end{bmatrix}$$

Show that the particle's velocity and acceleration can be described by the following vector equations, [1]

<u>Velocity</u>

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -5\pi\sin(\pi t) \\ 5\pi\cos(\pi t) \end{bmatrix}$$

<u>Acceleration</u>

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -5\pi^2\cos(\pi t) \\ -5\pi^2\sin(\pi t) \end{bmatrix}$$

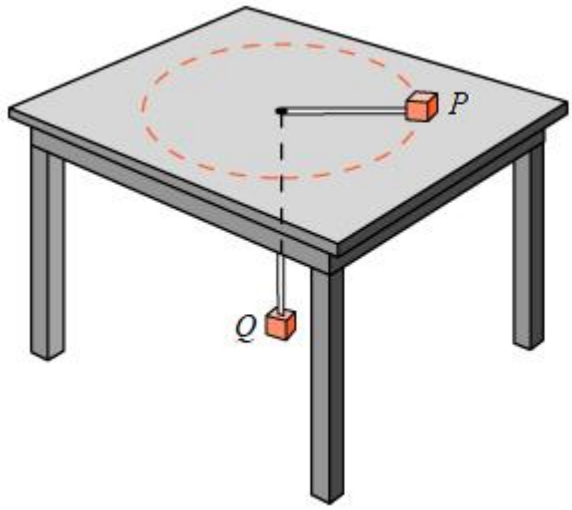
Part (b) continued…

(iii) Show that the equation of the path of the motion is a circle and find the radius and period of the motion. [3]

(iv) Describe the particle's acceleration vector relative to its position vector, at any time t, by referring to its direction and proportionality. [2]

(v) Find the dot product of the velocity and acceleration vectors. What does this imply about the motion? [2]

(vi) The particle P is moving on a smooth table and is attached to a second particle Q hanging below the table by a light string, as shown in the diagram. Taking gravity as $g = 10\ m/s^2$, find the mass of the second particle Q that is needed to allow for the motion of the first particle P. [2]



End of Question 15

Question 16 (15 marks)

Use a NEW sheet of paper.

(a)

(i) Show that, [3]

$$2\sin\theta\sum_{k=1}^{n}\sin 2k\theta = \cos\theta - \cos(2n+1)\theta$$

(ii) Hence, evaluate in exact form, [2]

$$2\sum_{k=1}^{302}\sin\frac{k\pi}{6}\cos\frac{k\pi}{6}$$

(b) Given that x, y and z are positive real numbers.

(i) Prove that $2\sqrt{xy} \leq x + y$. [1]

(ii) Hence, conclude that $8xyz \leq (x+y)(x+z)(y+z)$. [2]

(iii) Let a, b and c be the sides of a triangle. Show that [2]

$$(a+b-c)(a-b+c)(-a+b+c) \leq abc.$$

(c)

(i) Prove that $\sqrt{x}\left(1-\sqrt{x}\right)^{n-1} = \left(1-\sqrt{x}\right)^{n-1} - \left(1-\sqrt{x}\right)^{n}$ [1]

(ii) Let

$$I_n = \int_0^1 \left(1-\sqrt{x}\right)^n dx \quad \text{where } n = 1, 2, 3, \ldots$$

show that [3]

$$I_n = \frac{n}{n+2} I_{n-1}$$

(iii) Hence evaluate I_{100}. [1]

End of Question 16

End of Exam

Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet
Trial HSC Mathematics

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample 2 + 4 = ? (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct ↙

A ☒ B ☒ C ○ D ○

Student Number: 2020 Ext 2

Completely fill the response oval representing the most correct answer.

1. A ○ B ○ C ○ D ●
2. A ● B ○ C ○ D ○
3. A ○ B ● C ○ D ○
4. A ● B ○ C ○ D ○
5. A ○ B ○ C ○ D ●
6. A ● B ○ C ○ D ○
7. A ○ B ○ C ○ D ●
8. A ○ B ○ C ○ D ●
9. A ○ B ○ C ● D ○
10. A ○ B ○ C ● D ○

Question 11 (15 marks)

(a) If $z = 2 - i\sqrt{12}$

(i) Express z in modulus-argument form. [2]

(ii) Find the modulus and argument of z^5. [2]

i) $|z| = \sqrt{2^2 + (\sqrt{12})^2} = \sqrt{16} = 4 \quad \therefore |z| = 4$ (1)

$\arg z = \tan^{-1}\left(-\frac{\sqrt{12}}{2}\right)$

$= -\tan^{-1}(\sqrt{3})$

$= -\frac{\pi}{3}$ (1) $\quad \therefore z = 4\,\text{cis}\left(-\frac{\pi}{3}\right)$.

ii) $z^5 = r^5(\cos 5\theta + i\sin 5\theta)$

$|z^5| = |z|^5 = 4^5 = 1024$. (1)

$\arg(z^5) = -\frac{5\pi}{3} = \frac{\pi}{3}$ (1)

(b) Find

(i) [2]

$$\int \frac{2}{\sqrt{4-9x^2}}\,dx$$

(ii) [2]

$$\int \frac{2x}{\sqrt{4-9x^2}}\,dx$$

i) $\int \frac{2}{\sqrt{4-9x^2}}\,dx = \int \frac{2}{\sqrt{2^2-(3x)^2}}\,dx$

(1) (1)

$= \frac{2}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C$

ii) $\int \frac{2x}{\sqrt{4-9x^2}}\,dx$ let $u = x^2$

$du = 2x \cdot dx$

$= \int \frac{du}{\sqrt{4-9u}}$

$= \int (4-9u)^{-1/2}\,du$

$= \frac{2}{-9}(4-9u)^{1/2} + C$

(1) (1)

$= -\frac{2}{9}\sqrt{4-9x^2} + C$

OR $-\frac{1}{9}\int \frac{-2\times 9x}{\sqrt{4-9x^2}}\,dx$

$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$

$= -\frac{1}{9}\left[2\sqrt{4-9x^2}\right] + C$

$= -\frac{2}{9}\sqrt{4-9x^2} + C$

(c)

(i) Express $\frac{8-x}{x(x-2)^2}$ in the form $\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$, where A, B and C are constants. [2]

(ii) Hence find [2]

$$\int \frac{8-x}{x(x-2)^2}\,dx$$

i) $\frac{8-x}{x(x-2)^2} \equiv \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$

$8-x \equiv A(x-2)^2 + B(x)(x-2) + Cx$

when $x=0$: $8 = A(-2)^2 \quad \therefore A=2$

when $x=2$: $6 = 2C \quad \therefore C=3$ (1)

when $x=1$: $7 = A - B + C$

$7 = 2 - B + 3 \quad \therefore B=-2$.

$\therefore \frac{8-x}{x(x-2)^2} = \frac{2}{x} - \frac{2}{(x-2)} + \frac{3}{(x-2)^2}$ (1)

ii) $\int \frac{8-x}{x(x-2)^2}\,dx$

$= \int \left(\frac{2}{x} - \frac{2}{(x-2} + \frac{3}{(x-2)^2}\right) dx$

$= 2\ln|x| - 2\ln|x-2| + \int 3(x-2)^{-2}\,dx$ (1)

$= 2\ln\left|\frac{x}{x-2}\right| + \frac{3(x-2)^{-1}}{-1} + c$

$= 2\ln\left|\frac{x}{x-2}\right| - \frac{3}{(x-2)} + c$. (1)

loss of mark for negative sign being missed.

(d) If ω is a complex root of the equation $z^3 - 1 = 0$.

(i) Show that $1 + \omega + \omega^2 = 0$. [1]

(ii) Prove that $(a+b)(a+\omega b)(a+\omega^2 b) = a^3 + b^3$. [2]

i) $z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

since ω is a root then:

$(\omega - 1)(\omega^2 + \omega + 1) = 0$

① but $\omega \neq 1$, since ω is a complex root.

$\therefore \omega^2 + \omega + 1 = 0.$

☆ This question had to be shown carefully for one mark.

ii)

$LHS = (a+b)(a+wb)(a+w^2b)$

$= (a^2 + wab + ab + wb^2)(a + w^2 b)$

$= a^3 + w^2a^2b + wa^2b + w^3ab^2 + a^2b + w^2ab^2 + wab^2 + w^3b^3$ ① expansion

$= a^3 + a^2b(w^2 + w + 1) + ab^2(w^3 + w^2 + w) + w^3b^3$

$= a^3 + a^2b(0) + ab^2(1 + w + w^2) + (1)b^3$ ① justification.

$= a^3 + b^3$

$= RHS.$

Q12

a) $|z+2| < \frac{1}{3}$. Show $|6z+11| \le 3$

$$|6z+11| \le 3$$

$$|6(z+2)-1| \le 3 \checkmark$$

$$\left|6\times\frac{1}{3} - 1\right| \le 3$$

$$1 \le 3 \quad \text{True}$$

$$\therefore |6z+11| \le 3$$

Alternate Method

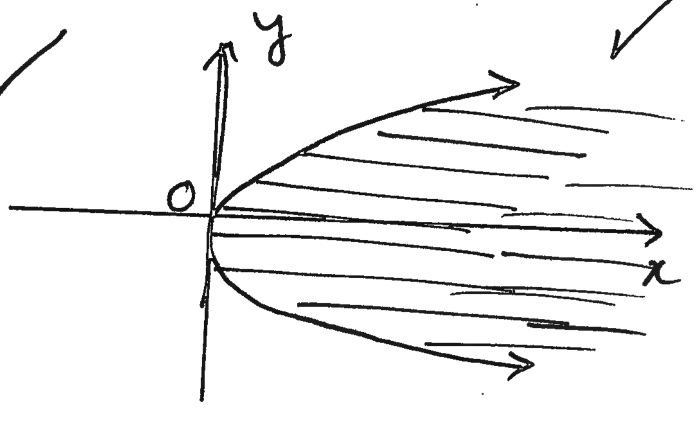
$$|6z+11| = |6z+12-1|$$
$$\le |6z+12| + |-1|$$
$$\le 6|z+2| + 1$$
$$\le 6\times\frac{1}{3} + 1$$
$$\le 3$$

✓ A number of students did not use the given result $|z+2|<\frac{1}{2}$. Hence didn't get the full mark.

b) $$\mathrm{Re}(z) \geqslant |z-\bar{z}|^2$$

$$\mathrm{Re}(z) \geqslant |2iy|^2 \checkmark$$

$$x \geqslant 4y^2 \checkmark$$



✓

Q12

c) $A(8, -19, 13)$ $B(7, -5, 10)$

i) $\begin{pmatrix} 7 \\ -15 \\ 10 \end{pmatrix} - \begin{pmatrix} 8 \\ -19 \\ 13 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$

Equation of the line through A and B.

$$r = \begin{pmatrix} 8 \\ -19 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$$

$$r = (8 - \lambda)\underset{\sim}{i} - (19 - 4\lambda)\underset{\sim}{j} + (13 - 3\lambda)\underset{\sim}{k}$$

ii) The two end points of the interval

when $\lambda = -3$: $\begin{pmatrix} 11 \\ -31 \\ 22 \end{pmatrix}$ when $\lambda = 7$: $\begin{pmatrix} 1 \\ 9 \\ -8 \end{pmatrix}$

Let the point divides the interval into the ratio of 3:2 be $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} a - 11 \\ b + 31 \\ c - 22 \end{pmatrix} = \frac{3}{5} \begin{pmatrix} 1 - 11 \\ 9 - -31 \\ -8 - 22 \end{pmatrix}$$

$a - 11 = \frac{3}{5} \times (-10)$

$b + 31 = \frac{3}{5}(40)$ $\therefore$

$c - 22 = \frac{3}{5}(-30)$

$a - 11 = -6$

$b + 31 = 24$ $\therefore$

$c - 22 = -18$

$a = 5$

$b = -7$

$c = 4$

Thus the point is $\begin{pmatrix} 5 \\ -7 \\ 4 \end{pmatrix}$.

Q12

d) i) $ax^3 + bx^2 + cx + d = p(x)$

as $x \to +\infty$ ∴ $p(x) \to +\infty$

as $x \to -\infty$ ∴ $p(x) \to -\infty$

Thus cubic polynomial will have at least one point of intersection with x-axis or cubic polynomial equation always has a real root. ✓

ii) If the coefficients are real (one pair of conjugate roots)

The roots are $\alpha i, -\alpha i, \beta$

· Sum of roots: $\alpha i - \alpha i + \beta = -b$

$\beta = -b$

· product of two roots at a time

$(\alpha i)(-\alpha i) + \alpha\beta i - \alpha\beta i = c$

$\alpha^2 = c$

· product of 3 roots

$(\alpha i)(-\alpha i)(\beta) = -d$ ✓

$\alpha^2 \beta = -d$

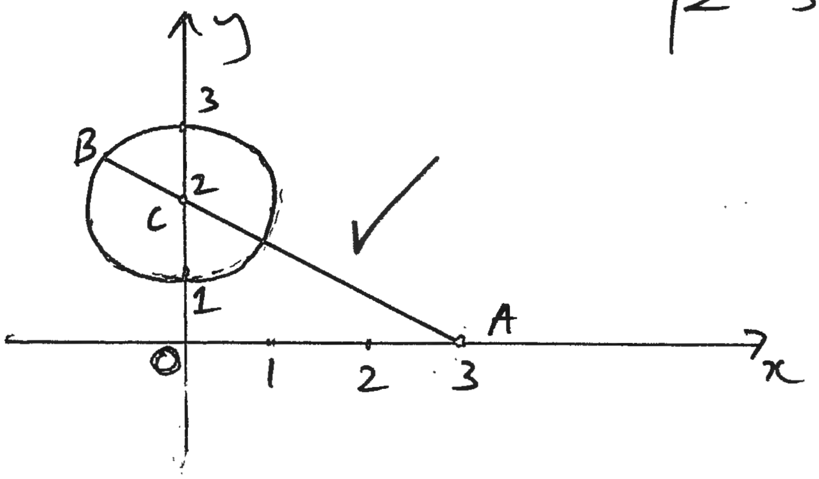
$c(-b) = -d$

$c = \frac{-d}{-b} > 0$ OR $c = \frac{d}{b}$ ✓ ∴ $\boxed{d = bc}$

Thus $\boxed{c > 0}$

Q12

e) $|z-2i| = 1$. Find the greatest value of $|z-3|$



$AC^2 = 2^2 + 3^2$

$AC = \sqrt{13}$

The Greatest value of $|z-3|$ must pass through the centre of the circle: $|z-2i| = 1$ (Radius = 1)

$\therefore$ Greatest $|z-3| = \sqrt{13} + 1$

Question 13

(a)(i) 6 is a perfect number and 12 is a multiple of 6. Factors of 12 are $\{1, 2, 3, 4, 6, 12\}$.

$$1 + 2 + 3 + 4 + 6 = 16 > 12$$

This counterexample disproves the conjecture since we have a multiple of a perfect number that isn't a perfect number.

> Some students didn't know the difference between a factor and a multiple.

(ii) p is a perfect number. Let the n factors of p (excluding p) in ascending order be $\{f_1, f_2, f_3, \dots, f_n\}$. Note that $f_1 = 1$ since it is a factor of all positive integers.

Let k be an integer such that $k \geq 2$. Assume that kp is a perfect number, thus

$$\begin{aligned} kp &= k(f_1 + f_2 + f_3 + \cdots + f_n) \\ &= kf_1 + kf_2 + kf_3 + \cdots + kf_n \end{aligned}$$

Note that $kf_1 > 1$ but 1 is a factor of kp so 1 must be included into the sum, thus

$$1 + kf_1 + kf_2 + kf_3 + \cdots + kf_n > kp$$

This contradicts our assumption, therefore kp is not perfect.

(b)(i) The extremes of $\frac{1}{5+3\cos x}$ will occur at the extremes of $\cos x$, that is -1 and 1.

Greatest

$$\frac{1}{5-3} = \frac{1}{2}$$

Least

$$\frac{1}{5+3} = \frac{1}{8}$$

(ii)

$$\frac{1}{8} \leq \frac{1}{5+3\cos x} \leq \frac{1}{2}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{8}\,dx \leq \int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x}\,dx \leq \int_0^{\frac{\pi}{2}} \frac{1}{2}\,dx$$

$$\left[\frac{1}{8}x\right]_0^{\frac{\pi}{2}} \leq \int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x}\,dx \leq \left[\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$$

$$\frac{\pi}{16} \leq \int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x}\,dx \leq \frac{\pi}{4}$$

(iii) $$\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x}\,dx = \int_0^1 \frac{1}{5+3\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{5+5t^2+3-3t^2}\,dt$$

$$= \int_0^1 \frac{1}{4+t^2}\,dt$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^1$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2}$$

$$\approx 0.232$$

> Pat (b) was generally done very well. Lots of students gave the final answer in degrees instead of radians.

(c)(i) $$z - z^{-1} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$

$$z - \frac{1}{z} = 2i\sin\theta$$

$$\frac{1}{2i}\left(z - \frac{1}{z}\right) = \sin\theta$$

(ii) $$z^n = \cos n\theta + i\sin n\theta$$

$$z^{-n} = \cos n\theta - i\sin n\theta$$

$$z^n - z^{-n} = 2i\sin n\theta$$

Most students did not show these results for n=3 and 5. Though they didn't lose marks for this in a similar question in the HSC they will. Given that part (i) was proving for n=1 these results cannot be assumed.

$\underline{n = 1}$

$$z - \frac{1}{z} = 2i\sin\theta$$

$\underline{n = 3}$

$$z^3 - \frac{1}{z^3} = 2i\sin 3\theta$$

$\underline{n = 5}$

$$z^5 - \frac{1}{z^5} = 2i\sin 5\theta$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\sin^5\theta = \left(\frac{1}{2i}\right)^5\left(z - \frac{1}{z}\right)^5$$

$$= \frac{1}{2^5 i}\left(z^5 - 5z^4\frac{1}{z} + 10z^3\frac{1}{z^2} - 10z^2\frac{1}{z^3} + 5z\frac{1}{z^4} - \frac{1}{z^5}\right)$$

$$= \frac{1}{2^5 i}\left(\left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)\right)$$

$$= \frac{1}{2^5 i}(2i\sin 5\theta - 5 \times 2i\sin 3\theta + 10 \times 2i\sin\theta)$$

$$\sin^5\theta = \frac{1}{2^4}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

(iii) $$\int \sin^5\theta\, d\theta = \frac{1}{16}\int(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)d\theta$$

$$= \frac{1}{16}\left(-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right) + C$$

Question 14

(a)(i) $MB = \frac{1}{2}|w - z|$

$$\tan\alpha = \frac{MP}{MB}$$

$$MP = MB\tan\alpha$$

$$MP = \frac{1}{2}|w - z|\tan\alpha$$

(ii) $\overrightarrow{AM} = \frac{1}{2}(w - z)$

$$\overrightarrow{MP} = i\frac{\overrightarrow{AM}}{|\overrightarrow{AM}|} \times |\overrightarrow{MP}|$$

$$= i\frac{\frac{1}{2}(w - z)}{MB} \times MP$$

$$= i\frac{\frac{1}{2}(w - z)}{\frac{1}{2}|w - z|} \times \frac{1}{2}|w - z|\tan\alpha$$

$$= \frac{1}{2}i(w - z)\tan\alpha$$

Many students did the 90° rotation by multiplying by i they did not show the scaling needed for the difference in lengths.

(iii) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AM} + \overrightarrow{MP}$

$$= z + \frac{1}{2}(w - z) + \frac{1}{2}i(w - z)$$

$$= z + \frac{1}{2}w - \frac{1}{2}z + \frac{1}{2}iw - \frac{1}{2}iz$$

$$= \frac{1}{2}w + \frac{1}{2}iw + \frac{1}{2}z - \frac{1}{2}iz$$

$$\overrightarrow{OP} = \frac{1}{2}(w + iw + z - iz)$$

(b) Prove for $n = 2$

$$RHS = \frac{4}{3} = 1 + \frac{1}{3}$$

$$LHS = \frac{1}{2} + \frac{2}{3} = \frac{7}{6} = 1 + \frac{1}{6} < RHS$$

Many students setting out for this induction was difficult to follow. Side results you need to prove your induction step should be done separately from the induction structure and then only referenced within the structure. To see how to reference results look at the solutions to the 2017 SGHS THSC question 16 (c).

Assume for $n = k$

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{k}{k+1} < \frac{k^2}{k+1}$$

Prove for $n = k + 1$

Required to prove

$$\frac{(k+1)^2}{k+2} - \left(\frac{1}{2} + \frac{2}{3} + \cdots + \frac{k}{k+1} + \frac{k+1}{k+2}\right) > 0$$

$$LHS > \frac{(k+1)^2}{k+2} - \left(\frac{k^2}{k+1} + \frac{k+1}{k+2}\right) \quad \text{by assumption}$$

$$= \frac{(k+1)^2}{k+2} - \frac{k^2}{k+1} - \frac{k+1}{k+2}$$

$$= \frac{(k+1)^3 - k^2(k+2) - (k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k^3 + 3k^2 + 3k + 1 - k^3 - 2k^2 - k^2 - 2k - 1}{(k+1)(k+2)}$$

$$= \frac{k}{(k+1)(k+2)}$$

$$> 0$$

Hence by the principles of mathematical induction the inequality is true for all integers $n \geq 2$.

(b) Another method

Prove for $n = 2$

$$RHS = \frac{4}{3} = 1 + \frac{1}{3}$$

$$LHS = \frac{1}{2} + \frac{2}{3} = \frac{7}{6} = 1 + \frac{1}{6} < RHS$$

Most students that gained full marks for this question used this method.

Assume for $n = k$

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{k}{k+1} < \frac{k^2}{k+1}$$

Prove for $n = k + 1$

Required to prove

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{k}{k+1} + \frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}$$

$$LHS < \frac{k^2}{k+1} + \frac{k+1}{k+2} \quad \text{by assumption}$$

$$= \frac{k^2(k+2) + (k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k^3 + 3k^2 + 2k + 1}{(k+1)(k+2)}$$

$$< \frac{k^3 + 3k^2 + 3k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^3}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{k+2}$$

$$= RHS$$

Hence by the principles of mathematical induction the inequality is true for all integers $n \geq 2$.

(c) $$\tan\alpha + \tan\beta + \tan\gamma = a + 1$$

$$\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\alpha\tan\gamma = c - a$$

$$\tan\alpha\tan\beta\tan\gamma = c$$

$$\tan(\alpha+\beta+\gamma) = \frac{\tan(\alpha+\beta) + \tan\gamma}{1 - \tan(\alpha+\beta)\tan\gamma}$$

$$= \frac{\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} + \tan\gamma}{1 - \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\tan\gamma}$$

$$= \frac{\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} + \dfrac{\tan\gamma\,(1 - \tan\alpha\tan\beta)}{1 - \tan\alpha\tan\beta}}{\dfrac{1 - \tan\alpha\tan\beta}{1 - \tan\alpha\tan\beta} - \dfrac{\tan\alpha\tan\gamma + \tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta}}$$

$$= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\alpha\tan\gamma - \tan\beta\tan\gamma}$$

$$= \frac{a + 1 - c}{1 - (c - a)}$$

$$= \frac{a + 1 - c}{1 - c + a}$$

$$= 1$$

$$\alpha + \beta + \gamma = \tan^{-1}(1)$$

$$= \frac{\pi}{4} + n\pi \quad \text{where } n \text{ is an integer}$$

(d)(i) Let $\underset{\sim}{c} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

$$\text{proj}_{\underset{\sim}{r}}\underset{\sim}{c} = \frac{\underset{\sim}{r} \cdot \underset{\sim}{c}}{\underset{\sim}{r} \cdot \underset{\sim}{r}}\underset{\sim}{r}$$

$$= \frac{x_0 + 2y_0 + z_0}{1^2 + 2^2 + 1^2}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{x_0 + 2y_0 + z_0}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(ii) $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 4 - t \\ 1 + t \\ 5t \end{bmatrix}$

$$\text{proj}_{\underset{\sim}{r}}\underset{\sim}{c} = \frac{4 - t + 2 + 2t + 5t}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{6t + 6}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= (t + 1)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

So the point on $\underset{\sim}{r}$ that has a perpendicular distance to a point on $\underset{\sim}{c}$ is $(t + 1, 2t + 2, t + 1)$ for any t. And that point on $\underset{\sim}{c}$ is $(4 - t, 1 + t, 5t)$.

Many students found these two points but then didn't go onto find the t value that minimize the distance between them.

Let l be the distance between the two points, so

$$l^2 = (4 - t - t - 1)^2 + (1 + t - 2t - 2)^2 + (5t - t - 1)^2$$

$$l^2 = (3 - 2t)^2 + (-t - 1)^2 + (4t - 1)^2$$

$$l^2 = 4t^2 - 12t + 9 + t^2 + 2t + 1 + 16t^2 - 8t + 1$$

$$l^2 = 21t^2 - 18t + 11$$

$$\frac{dl^2}{dt} = 42t - 18$$

Stationary point

$$42t - 18 = 0$$

$$t = \frac{3}{7}$$

$$\frac{d^2l^2}{dt^2} = 42 > 0 \text{ minima}$$

The position vector is

$$\left(\frac{3}{7} + 1\right)\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$= \left(\frac{10}{7}\right)\begin{bmatrix}1\\2\\1\end{bmatrix}$$

So the point is $\left(\frac{10}{7}, \frac{20}{7}, \frac{10}{7}\right)$

Question 15 (a)

(i) $u = a - x \qquad du = -dx$

when $x = a$, $u = a - a$
$= 0$

$x = 0$, $u = a - 0$
$= a$

$$\therefore \int_0^a x f(x)\,dx = \int_0^a x f(a-x)\,dx \quad \text{given } f(x) = f(a-x)$$

$$= \int_a^0 (a-u) f(u) \times -du$$

$$= \int_0^a (a-u) f(u)\,du$$

$$= \int_0^a (a-x) f(x)\,dx$$

$$= \int_0^a a f(x)\,dx - \int_0^a x f(x)\,dx$$

$$\therefore 2\int_0^a x f(x)\,dx = a\int_0^a f(x)\,dx$$

Hence $\displaystyle\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$

The quality of the proofs varied.
Many students need to show the steps
more carefully and clearly.

Question 15 (a)

(ii) Let $f(x) = \dfrac{\sin x}{1+\cos^2 x}$

$$f(\pi - x) = \frac{\sin(\pi - x)}{1+(\cos(\pi - x))^2}$$

$$= \frac{\sin x}{1+(-\cos x)^2}$$

$$= \frac{\sin x}{1+\cos^2 x} = f(x)$$

The majority of students did not check that the (i) result could be used. This working should have been included.

Hence, we can use the result from (a)(i).

$$\int_0^{\pi} x \times \frac{\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

Most students evaluated the integral correctly.

$$= \frac{\pi}{2}\int_{\cos 0}^{\cos \pi} \frac{-du}{1+u^2}$$

$$= \frac{\pi}{2}\Big[-\tan^{-1} u\Big]_1^{-1}$$

$$= \frac{\pi}{2}\left(-\tan^{-1}(-1) + \tan^{-1}(1)\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} \times \frac{\pi}{2}$$

$$\therefore I = \frac{\pi^2}{4}$$

Question 15 (b)

(i)

$x = 5\cos(\pi t)$

$\dot{x} = -5\sin(\pi t) \times \pi$

$\dot{x}^2 = (-5\pi \sin(\pi t))^2$

$= 25\pi^2 \sin^2(\pi t)$

$= 25\pi^2(1 - \cos^2(\pi t))$

$= 25\pi^2 - \pi^2 \times 25\cos^2(\pi t)$

$\therefore \dot{x}^2 = 25\pi^2 - \pi^2 x^2$ as required.

Different approaches used but the question is expecting a substitution approach into the differential equation, rather than using $v^2 = n^2(a^2 - x^2)$.

(ii)

$x = 5\cos(\pi t)$

$\dot{x} = \frac{d}{dt}(5\cos(\pi t))$

$= -5\sin(\pi t) \times \pi$

$= -5\pi \sin(\pi t)$

$\ddot{x} = \frac{d}{dt}(-5\pi \sin(\pi t))$

$= -5\pi \cos(\pi t) \times \pi$

$= -5\pi^2 \cos(\pi t)$

$y = 5\sin(\pi t)$

$\dot{y} = \frac{d}{dt}(5\sin(\pi t))$

$= 5\cos(\pi t) \times \pi$

$= 5\pi \cos(\pi t)$

$\ddot{y} = \frac{d}{dt}(5\pi \cos(\pi t))$

$= -5\pi \sin(\pi t) \times \pi$

$= -5\pi^2 \sin(\pi t)$

$$\therefore \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -5\pi \sin(\pi t) \\ 5\pi \cos(\pi t) \end{bmatrix}$$

$$\text{and } \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} -5\pi^2 \cos(\pi t) \\ -5\pi^2 \sin(\pi t) \end{bmatrix}$$

Students were expected to show how the acceleration and velocity vectors were obtained. Writing only what was given in the question is not sufficient.

Question 15 (b)

(iii) $x^2 + y^2 = (5\cos(\pi t))^2 + (5\sin(\pi t))^2$

$= 25\cos^2(\pi t) + 25\sin^2(\pi t)$

$= 25$ since $\cos^2(\pi t) + \sin^2(\pi t) = 1$

$\therefore$ The path of the motion is a circle with radius 5 m.

Given $v^2 = n^2(a^2 - x^2)$ for simple harmonic motion

and $\dot{x}^2 = \pi^2(25 - x^2)$, $n = \pi$

and the period of motion $= \frac{2\pi}{\pi} = 2$ seconds.

(iv) Observe $\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -\pi^2 \begin{bmatrix} x \\ y \end{bmatrix}$.

$\therefore$ The particle's acceleration is in the opposite direction to its position vector at any time. The magnitude of the acceleration is π^2 times that of the magnitude of the displacement, i.e. the magnitude is $5\pi^2$ m/s^2 at any time.

Many answers were not phrased carefully enough. Stating the acceleration was negative did not provide enough of a sense of the relationship with the position vector.

Question 15 (b)

(v) $\underset{\sim}{v} \cdot \underset{\sim}{a} = -5\pi \sin(\pi t) \times -5\pi^2 \cos(\pi t) + 5\pi \cos(\pi t) \times -5\pi^2 \sin(\pi t)$

$= 25\pi^3 \sin(\pi t)\cos(\pi t) - 25\pi^3 \sin(\pi t)\cos(\pi t)$

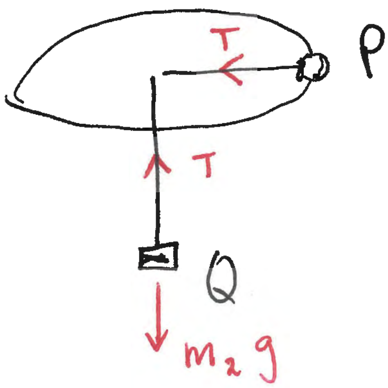
$= 0$

$\therefore$ Velocity and acceleration are perpendicular at all times.

Since the acceleration is directed towards the centre, the velocity will be in the direction of the tangent.

(vi) For the particle at P, $|\underset{\sim}{a}| = 5\pi^2$.

In order for the system to stay as described:



$T = m_2 g$

$T = m_1 a$ where $m_1 = 3$

i.e. $10 m_2 = 3 \times 5\pi^2$

$\therefore m_2 = \dfrac{3\pi^2}{2}$ kg

This question was not done well as a whole. Students needed to consider the acceleration of the particle (in circular motion) and the tension in the string.

Question 16 (15 marks)

Use a NEW sheet of paper.

(a)

(i) Show that, [3]

$$2\sin\theta\sum_{k=1}^{n}\sin 2k\theta = \cos\theta - \cos(2n+1)\theta$$

$2\sin\theta \sum_{k=1}^{n} \sin 2k\theta$

$= 2\sin\theta\left[\sin 2\theta + \sin 4\theta + \sin 6\theta + \dots + \sin 2(n-1)\theta + \sin 2n\theta\right]$

$= 2\left[\sin\theta\sin 2\theta + \sin\theta\sin 4\theta + \sin\theta\sin 6\theta + \dots + \sin\theta\sin(2n-2)\theta + \sin\theta\sin 2n\theta\right]$ ① expansion

(using products to sums ... reference sheet)

$= 2\times\frac{1}{2}\left[\cos(2\theta-\theta) - \cos(2\theta+\theta) + \cos(4\theta-\theta) - \cos(4\theta+\theta) + \cos(6\theta-\theta) - \cos(6\theta+\theta) + \dots + \cos(2n\theta-2\theta-\theta) - \cos(2n\theta-2\theta+\theta) + \cos(2n\theta-\theta) - \cos(2n\theta+\theta)\right]$

① show enough terms to develop a pattern.

$= \cos\theta - \cos 3\theta + \cos 3\theta - \cos 5\theta + \cos 5\theta - \cos 7\theta + \dots + \cos(2n\theta - 3\theta) - \cos(2n\theta-\theta) + \cos(2n\theta-\theta) - \cos(2n\theta+\theta)$

by symmetry, all terms cancel out except for the first and last term.

$= \cos\theta - \cos(2n\theta+\theta)$

① Justifying cancelling out of terms..

$= \cos\theta - \cos(2n+1)\theta.$

☆ Part a) Was challenging for most students. Steps needed to be shown clearly for full marks.

Alternative proof: by induction for $n \in \mathbb{Z}^+, n \geq 1$.

When $n=1$: $RHS = \cos\theta - \cos(2(1)+1)\theta$

$= \cos\theta - \cos 3\theta$.

$LHS = 2\sin\theta \sum_{k=1}^{1} \sin 2k\theta$

$= 2\sin\theta \sin 2\theta$

$= \cos(2\theta-\theta) - \cos(2\theta+\theta)$

$= \cos\theta - \cos 3\theta$

$= RHS$. ① LHS = RHS.

$\therefore$ true for $n=1$.

Assume true for $n=a$, $a \in \mathbb{Z}^+$, $a \geq 1$.

i.e. $2\sin\theta \sum_{k=1}^{a} \sin 2k\theta = \cos\theta - \cos(2a+1)\theta$ .. ①

Prove true for $n=a+1$.

i.e. Show $2\sin\theta \sum_{k=1}^{(a+1)} \sin 2k\theta = \cos\theta - \cos(2a+3)\theta$

Proof: $LHS = 2\sin\theta \sum_{k=1}^{(a+1)} \sin 2k\theta$ ① set-up with assumption

$= 2\sin\theta \left[\sum_{k=1}^{a} \sin 2k\theta + \sin 2(a+1)\theta \right]$

$= 2\sin\theta \sum_{k=1}^{a} \sin 2k\theta + 2\sin\theta \sin 2(a+1)\theta$

(by assumption)

$= \cos\theta - \cos(2a+1)\theta + \cos(2a+2-1)\theta - \cos(2a+2+1)\theta$

$= \cos\theta - \cos(2a+1)\theta + \cos(2a+1)\theta - \cos(2a+3)\theta$

$= \cos\theta - \cos(2a+3)\theta$ ① clear steps in solution.

$= RHS$.

$\therefore$ true for $n=a+1$ if true for $n=a$.

$\therefore$ By Principle of Mathematical Induction true for $n \geq 1$, $n \in \mathbb{Z}^+$.

(ii) Hence, evaluate in exact form,

$$2\sum_{k=1}^{302} \sin\frac{k\pi}{6}\cos\frac{k\pi}{6}$$ [2]

$= \sum_{k=1}^{302} 2\sin\frac{k\pi}{6}\cos\frac{k\pi}{6}$

$= \sum_{k=1}^{302} \sin 2k\left(\frac{\pi}{6}\right)$

From part i) with $\theta = \frac{\pi}{6}$, $n = 302$: ① show connection with part(i)

$\Rightarrow 2\sin\frac{\pi}{6} \sum_{k=1}^{302} \sin 2k\left(\frac{\pi}{6}\right)$

$= \cos\frac{\pi}{6} - \cos\left(2\times 302+1\right)\frac{\pi}{6}$

$= \cos\frac{\pi}{6} - \cos\left(\frac{600\pi}{6} + \frac{5\pi}{6}\right)$

$= \cos\frac{\pi}{6} - \cos\frac{5\pi}{6}$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \sqrt{3}$. ① correct answer.

★ (This part had to be related to part i) for the correct answer.)

(b) Given that x, y and z are positive real numbers.

(i) Prove that $2\sqrt{xy} \leq x + y$. [1]

(ii) Hence, conclude that $8xyz \leq (x+y)(x+z)(y+z)$. [2]

(iii) Let a, b and c be the sides of a triangle. Show that [2]

$$(a+b-c)(a-b+c)(-a+b+c) \leq abc.$$

i) Many methods ...

$$(\sqrt{x} - \sqrt{y})^2 \geq 0$$

$$x - 2\sqrt{xy} + y \geq 0$$

$$x + y \geq 2\sqrt{xy}$$

$$\therefore \quad 2\sqrt{xy} \leq x + y, \text{ as required.}$$ (1)

Well done

ii) Similarly, using part i):

$$2\sqrt{xy} \leq (x+y)$$

$$2\sqrt{xz} \leq (x+z)$$

$$2\sqrt{yz} \leq (y+z)$$ (1)

Well done

Hence:

$$2\sqrt{xy} . 2\sqrt{xz} . 2\sqrt{yz} \leq (x+y)(x+z)(y+z)$$

$$8\sqrt{x^2y^2z^2} \leq (x+y)(x+z)(y+z)$$

$$\therefore 8xyz \leq (x+y)(x+z)(y+z).$$ (1)

iv) By the triangular inequality, the sum of two sides of a triangle is greater than or equal to the third side.

$$\therefore a+b \geq c \Rightarrow a+b-c \geq 0 \Rightarrow x$$

$$a+c \geq b \Rightarrow a-b+c \geq 0 \Rightarrow y$$

$$b+c \geq a \Rightarrow -a+b+c \geq 0 \Rightarrow z.$$

P.T.O.

From part ii) $8xyz \le (x+y)(x+z)(y+z)$.

$RHS = (x+y)(x+z)(y+z)$

$= (a+b-c+a-b+c)(a+b-c-a+b+c)(a-b+c-a+b+c)$

$= 2a \times 2b \times 2c$ (1)

$= 8abc$

$LHS = 8xyz$

$= 8(a+b-c)(a-b+c)(-a+b+c)$

Hence $LHS \le RHS$

$\therefore 8(a+b-c)(a-b+c)(-a+b+c) \le 8abc$ (1)

i.e. $(a+b-c)(a-b+c)(-a+b+c) \le abc$, as required.

★ Other solutions attempted, need to show proof clearly for 2 marks.

(c)

(i) Prove that $\sqrt{x}(1-\sqrt{x})^{n-1} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^{n}$ [1]

(ii) Let

$$I_n = \int_0^1 (1-\sqrt{x})^n dx \quad \text{where } n = 1, 2, 3, \ldots$$

show that [3]

$$I_n = \frac{n}{n+2} I_{n-1}$$

(iii) Hence evaluate I_{100}. [1]

i) $RHS = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^{n}$

$= (1-\sqrt{x})^{n-1}[1-(1-\sqrt{x})]$ ①

$= (1-\sqrt{x})^{n-1}[1-1+\sqrt{x}]$

$= \sqrt{x}(1-\sqrt{x})^{n-1}$ ☆ Easiest approach!

$= LHS.$

OR

$LHS = \sqrt{x}(1-\sqrt{x})^{n-1}$

$= -[(1-\sqrt{x})-1](1-\sqrt{x})^{n-1}$

$= -[(1-\sqrt{x})^{n} - (1-\sqrt{x})^{n-1}]$

$= (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^{n}$

$= RHS.$

ii) $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$

let $u = (1-\sqrt{x})^n$ $\quad dv = dx$

$v = x.$

$du = n(1-\sqrt{x})^{n-1} \times \frac{-1}{2\sqrt{x}} \, dx$

$\therefore I_n = uv - \int v \, du$ (1).

$= \left[x(1-\sqrt{x})^n\right]_0^1 - \int_0^1 \frac{nx(1-\sqrt{x})^{n-1}}{-2\sqrt{x}} \, dx$

$= (0-0) + \frac{n}{2}\int_0^1 \sqrt{x}(1-\sqrt{x})^{n-1} \, dx.$

(from part i) $\Rightarrow \frac{n}{2}\int_0^1 \left[(1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n\right] dx.$ (1)

$= \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1} \, dx - \frac{n}{2}\int_0^1 (1-\sqrt{x})^n \, dx$

$\therefore I_n = \frac{n}{2} I_{n-1} - \frac{n}{2} I_n$

$I_n + \frac{n}{2} I_n = \frac{n}{2} I_{n-1}$

$\left(\frac{2+n}{2}\right) I_n = \frac{n}{2} I_{n-1}$

$\therefore I_n = \frac{n}{2} \times \frac{2}{2+n} I_{n-1}$ (1).

$I_n = \frac{n}{n+2} I_{n-1}$, as required.

* Many students who did not use I.B.P as the first step, were awarded only one mark.

iii) $I_n = \frac{n}{n+2} I_{n-1} \quad (n \geq 1)$

$I_1 = \frac{1}{1+2} I_0$

$= \frac{1}{3} \times \int_0^1 (1-\sqrt{x})^0 \, dx.$

$= \frac{1}{3} \times [x]_0^1$

$= \frac{1}{3} \times 1$

$= \frac{1}{3}.$ (That is: $I_0 = 1$).

$I_{100} = \frac{100}{102} I_{99}$

$= \frac{100}{102} \cdot \frac{99}{101} I_{98}$

$= \frac{100}{102} \cdot \frac{99}{101} \cdot \frac{98}{100} \cdot I_{97}$

$= \frac{100 \times 99 \times 98 \times 97 \times \dots \times 4 \times 3 \times 2}{102 \times 101 \times 100 \times \dots \times 4} \times I_1$

$= \frac{3 \times 2}{102 \times 101} \times \frac{1}{3}$

$= \frac{1}{51 \times 101}$

$= \frac{1}{5151}$ ① Correct answer.